Hilbert Transforms on $L^p(\mathbb{R})$

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1   Introduction

The Hilbert Transform had its origin in solving the following problem: given a real-valued function $f : \mathbb{R} \to \mathbb{R}$, can we find another function $g : \mathbb{R} \to \mathbb{R}$ such that the complex function $f + ig$ can be analytically extended to the upper half plane? Using methods of complex analysis, this can be achieved by finding the poisson integral of the given function, taking the harmonic conjugate of the integral in the upper half-plane and obtaining the boundary value function of the harmonic conjugate function [3]. This question may also be conversely posed as: what is the relationship between the boundary values of the real and imaginary parts of an analytic function [4]?

Hardy gave a formal definition in [2] to the Hilbert transform. While Hilbert transforms are defined for functions on $\mathbb{R}$, they provide motivation for singular integral operators defined for functions on $\mathbb{R}^n$ [4]. Much of the theory of singular integrals was developed by Zygmund and Calderon (See [3], [5]), and this theory forms the basis of modern harmonic analysis.

The idea that a function and its Hilbert transform can be used as real and imaginary parts of an analytic function is exploited in [6], [7], [8] to propose an efficient transform called Complex Wavelet Transform for signal processing applications. While (traditional) wavelet transforms have proved to be very successful in the sphere of signal processing, they suffer from problems such as shift variance [6], [7]. Such problems are not seen by complex wavelet transforms [6]. In case of complex wavelet transform, two wavelet functions

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are used as the real and imaginary parts of a complex function where the imaginary part is the Hilbert transform of the real part.

In this paper, we use the definition of Hilbert transforms given by Hardy [2]. We study the existence and properties of Hilbert transforms on $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Most of the technical material in this paper is based on [5]. In presentation of material in this paper, we assume knowledge of real analysis including Lebesgue integration, $L^p(\mathbb{R})$ spaces and the results involving them such as the Hölder's inequality and Minkowski's inequality. In addition, we assume knowledge of Fourier analysis. We give an appendix at the end stating the definitions and theorems of Fourier analysis that we use in this paper.

Definition 1.1. *Hilbert transform, denoted $\tilde{f}$ or $H(f)$, of a function $f : \mathbb{R} \to \mathbb{C}$ is defined as its convolution with function $\frac{1}{t}$ taken in the principal value sense:*

$$\tilde{f}(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{|x-t|>\varepsilon} \frac{f(t)}{x-t} \, dt$$

We use the notations $H(f)$ and $\tilde{f}$ interchangeably in this paper.

To analyze the properties of Hilbert transforms, we first define:

$$\tilde{f}_\varepsilon(x) = \frac{1}{\pi} \int\limits_{|x-t|>\varepsilon} \frac{f(t)}{x-t} \, dt$$

$$\tilde{f}_{\varepsilon,\omega}(x) = \frac{1}{\pi} \int\limits_{\varepsilon<|x-t|<\omega} \frac{f(t)}{x-t} \, dt$$

Now, for each $x \in \mathbb{R}$, the function in t, $\frac{1}{x-t} \in L^q((-\infty, x - \varepsilon) \cup (x + \varepsilon, \infty)), 1 < q \leq \infty$. Hence, by Hölder's inequality, for $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, $\tilde{f}_\varepsilon(x)$ exists for each $x \in \mathbb{R}$. This implies $\tilde{f}_{\varepsilon,\omega}(x)$ also exists for each $x \in \mathbb{R}$.

We now analyze the Hilbert transforms on $L^2(\mathbb{R})$.

2   Hilbert transforms on $L^2(\mathbb{R})$

Theorem 2.1. *Let $f \in L^2(\mathbb{R})$ and $0 < \varepsilon < \omega < \infty$. Then,*

  (i) *there exists $A > 0$ independent of f, such that $\|\tilde{f}_{\varepsilon,\omega}\|_2 \le A\|f\|_2$*

  (ii) *there exists $\tilde{f} \in L^2(\mathbb{R})$ such that $\|\tilde{f} - \tilde{f}_{\varepsilon,\omega}\|_2 \to 0$ as $\varepsilon \to 0$ and $\omega \to \infty$ simultaneously or successively.*

  (iii) $\|f\|_2 = \|\tilde{f}\|_2$

  (iv) $\tilde{\tilde{f}} = -f$

  (v) $\tilde{f}_\varepsilon(x) \to \tilde{f}(x)$ *pointwise a.e.*

Proof. Define

$$K_{\varepsilon,\omega}(t) = \begin{cases} \frac{2}{t} & \text{if } \varepsilon < |t| < \omega \\ 0 & \text{otherwise} \end{cases}$$

And, we observe

$$\tilde{f}_{\varepsilon,\omega}(x) = \frac{1}{\pi} \int\limits_{\varepsilon < |x-t| < \omega} \frac{f(t)}{x - t}\, dt$$

$$= \frac{1}{2\pi} \int\limits_{\varepsilon < |x-t| < \omega} \frac{2f(t)}{x - t}\, dt$$

$$= (f * K_{\varepsilon,\omega})(x)$$

Here, $*$ indicates convolution operation. Refer to (Definition 6.1).

Taking Fourier transforms on both sides and using (Theorem 6.9), we have:

$$\hat{\tilde{f}}_{\varepsilon,\omega} = \hat{f}\hat{K}_{\varepsilon,\omega} \text{ pointwise a.e.} \tag{1}$$

Now,

$$\hat{K}_{\varepsilon,\omega}(\gamma) = \frac{1}{2\pi} \int\limits_{\varepsilon < |t| < \omega} \frac{2}{t} e^{i\gamma t} \, dt$$

$$= \frac{1}{\pi} \int\limits_{\varepsilon}^{\omega} \frac{e^{-i\gamma t} - e^{i\gamma t}}{t} \, dt$$

$$= \frac{-2i}{\pi} \int\limits_{\varepsilon}^{\omega} \frac{\sin \gamma t}{t} \, dt$$

$$= \frac{-2i}{\pi} \int\limits_{\varepsilon\gamma}^{\omega\gamma} \frac{\sin u}{u/\gamma} \frac{du}{\gamma}$$

$$= \frac{-2i(\operatorname{sgn}\gamma)}{\pi} \int\limits_{\varepsilon|\gamma|}^{\omega|\gamma|} \frac{\sin u}{u} \, du \tag{2}$$

From section 1.6.3 of [1], we know

$$\int\limits_{0}^{\infty} \frac{\sin u}{u} \, dt = \frac{\pi}{2} \tag{3}$$

Thus, for every $\gamma, \varepsilon, \omega$, we can find $A > 0$ such that $|\hat{K}_{\varepsilon,\omega}(\gamma)| < A$. Hence, from (1), we have $\|\hat{\tilde{f}}_{\varepsilon,\omega}\|_2 < A\|\hat{f}\|_2$.

Then, from Parseval's formula (Theorem 6.8), we have that $\|\tilde{f}_{\varepsilon,\omega}\|_2 < A\|f\|_2$.

From (1), (2) and (3), we have that

$$\hat{\tilde{f}}_{\varepsilon,\omega}(\gamma) \to -i(\operatorname{sgn}\gamma)\hat{f}(\gamma) \text{ in } L^2 \tag{4}$$

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Then, Plancheral Theorem (Theorem 6.8) implies that there exists $\tilde{f} \in L^2(\mathbb{R})$ such that

$$\hat{\tilde{f}}(\gamma) = -i(\operatorname{sgn}\gamma)\hat{f}(\gamma) \tag{5}$$

and

$$\tilde{f}_{\varepsilon,\omega} \to \tilde{f} \text{ in } L^2 \tag{6}$$

From (5), we also see that $\|\hat{\tilde{f}}\|_2 = \|\hat{f}\|_2$. Again, by Parseval's formula, this implies $\|\tilde{f}\|_2 = \|f\|_2$.

Further, using (5), we see that $\hat{\tilde{\tilde{f}}}(x) = -i(\operatorname{sgn}x)\hat{\tilde{f}}(x) = (-i(\operatorname{sgn}x))^2\hat{f}(x) = -\hat{f}(x)$. Because the Fourier transform on $L^2(\mathbb{R})$ is a linear bijection (Theorem 6.8), this implies $\tilde{\tilde{f}} = -f$.

It remains to prove $\tilde{f}_\varepsilon(x) \to \tilde{f}(x)$ pointwise a.e. as $\varepsilon \to 0$. For this, the proof consists of showing that $(f * \tilde{P}_\varepsilon)(x) \to \tilde{f}(x)$ pointwise a.e. as $\varepsilon \to 0$ and $f_\varepsilon(x) - (f * \tilde{P}_\varepsilon)(x) \to 0$ pointwise a.e. as $\varepsilon \to 0$. Here, $\tilde{P}(x) = \frac{1}{\pi}\frac{x}{1+x^2}$ is the conjugate Poisson kernel and $\tilde{P}_\varepsilon(x) = \frac{1}{\varepsilon}\tilde{P}(\frac{x}{\varepsilon}) = \frac{1}{\pi}\frac{x}{\varepsilon^2+x^2}$. We suggest the reader to refer to [5] for details of the proof.

$\square$

The following properties follow easily from the definition and the above theorem:

(i) $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a continuous linear functional.

(ii) H is a bijection with the inversion formula $f(x) = -\frac{1}{\pi}\lim_{\varepsilon \to 0}\int_{|x-t|>\varepsilon} \frac{\tilde{f}(t)}{x-t}\,dt$.

(iii) $\overline{\tilde{f}} = \tilde{\overline{f}}$. That is, Hilbert transform commutes with conjugation. In particular, Hilbert transform of a real-valued function is real-valued.

(iv) $H(f(x+a)) = H(f)(x+a)$. That is, Hilbert transform commutes with translations.

The following corollarly captures additional properties of Hilbert transforms on $L^2(\mathbb{R})$:

Corollary 2.2. *Let $f, g \in L^2(\mathbb{R})$. Then,*

(i) $\int f\bar{g}\,dx = \int \tilde{f}\bar{\tilde{g}}\,dx$

(ii) $\int fg\,dx = \int \tilde{f}\tilde{g}\,dx$

(iii) $\int \tilde{f}g\,dx = -\int f\tilde{g}\,dx$

Proof. $|f+g|^2 = (f+g)(\bar{f}+\bar{g}) = |f|^2 + |g|^2 + f\bar{g} + \bar{f}g = |f|^2 + |g|^2 + 2\,\mathrm{Re}(f\bar{g})$

Similarly, $|H(f+g)|^2 = |\tilde{f}+\tilde{g}|^2 = |\tilde{f}|^2 + |\tilde{g}|^2 + 2\,\mathrm{Re}(\tilde{f}\bar{\tilde{g}})$

Then, from (Theorem 2.1), since $\|f+g\|_2 = \|H(f+g)\|_2$, we have that:
$\int_{\mathbb{R}} |f|^2 + |g|^2 + 2\,\mathrm{Re}(f\bar{g})\,dx = \int_{\mathbb{R}} |\tilde{f}|^2 + |\tilde{g}|^2 + 2\,\mathrm{Re}(\tilde{f}\bar{\tilde{g}})\,dx$

Again, from from (Theorem 2.1), since $\|f\|_2 = \|\tilde{f}\|_2$ and $\|g\|_2 = \|\tilde{g}\|_2$, we have that:

$$\mathrm{Re}\int_{\mathbb{R}} f\bar{g}\,dx = \mathrm{Re}\int_{\mathbb{R}} \tilde{f}\bar{\tilde{g}}\,dx \tag{7}$$

Replacing f by if in the above argument, we have

$$\mathrm{Im}\int_{\mathbb{R}} f\bar{g}\,dx = \mathrm{Im}\int_{\mathbb{R}} \tilde{f}\bar{\tilde{g}}\,dx \tag{8}$$

Combining (7) and (8), we have

$$\int f\bar{g}\,dx = \int \tilde{f}\bar{\tilde{g}}\,dx \tag{9}$$

In (9), using $\bar{g}$ in place of g and using the property $\bar{\bar{g}} = \tilde{g}$, we have

$$\int fg\,dx = \int \tilde{f}\tilde{g}\,dx \tag{10}$$

In (10), using $\tilde{f}$ in place of f and using the property $\tilde{\tilde{f}} = -f$, we have
$\int \tilde{f}g\,dx = \int \tilde{\tilde{f}}\tilde{g}\,dx = -\int f\tilde{g}\,dx.$ $\qquad\square$

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As an example, the Hilbert transform of the Poisson kernel, namely $P_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+x^2}$, is the conjugate Poisson kernel $\frac{1}{\pi}\frac{x}{\varepsilon^2+x^2}$, rightly denoted as $\tilde{P}_\varepsilon(x)$.

Corollary 2.3. *Let $f \in L^2(\mathbb{R})$. Let $P_\varepsilon(x)$ and $\tilde{P}_\varepsilon(x)$ denote the Poisson kernel $\frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2+x^2}$ and the conjugate Poisson kernel $\frac{1}{\pi}\frac{x}{\varepsilon^2+x^2}$ respectively. Then, $f * \tilde{P}_\varepsilon = \tilde{f} * P_\varepsilon$.*

Proof. By (Theorem 2.1), $\tilde{f} \in L^2(\mathbb{R})$.

Since $P_\varepsilon \in L^1(\mathbb{R})$, by Young's theorem (Theorem 6.3), $\tilde{f} * P_\varepsilon \in L^2(\mathbb{R})$.

Further, using (Corollary 2.2), and noting that $P_\varepsilon(x)$ is even and $\tilde{P}_\varepsilon(x)$ is odd, we have:

$$
\begin{aligned}
(\tilde{f} * P_\varepsilon)(x) &= \int_\mathbb{R} \tilde{f}(x-t)P_\varepsilon(t)\, dt \\
&= \int_\mathbb{R} \tilde{f}(x+t)P_\varepsilon(t)\, dt \\
&= -\int_\mathbb{R} f(x+t)\tilde{P}_\varepsilon(t)\, dt \\
&= \int_\mathbb{R} f(x-t)\tilde{P}_\varepsilon(t)\, dt \\
&= (f * \tilde{P}_\varepsilon)(x)
\end{aligned}
$$

$\square$

3   Hilbert transforms on $L^p(\mathbb{R})$ for $1 < p < \infty$

We consider the function $F(z) = \frac{1}{\pi i}\int\limits_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt$ defined for complex arguments $z = x + iy$ in the upper half plane$(y = \text{Im}(z) > 0)$.

Writing $\frac{1}{t-z}$ as $\frac{iy}{(t-x)^2+y^2} - \frac{x-t}{(t-x)^2+y^2}$, we have $\frac{1}{\pi i}\frac{1}{t-z} = P_y(x-t)+i\tilde{P}_y(x-t)$, where $P_y(x)$ is the Poisson kernel and $\tilde{P}_y$ is the conjugate Poisson kernel.

Thus, $F(z) = F(x+iy) = (f * P_y)(x) + i(f * \tilde{P}_y)(x)$. And, we define $U(x,y) = (f * P_y)(x)$ and $V(x,y) = (f * \tilde{P}_y)(x)$.

With the above definitions, we develop the theory of Hilbert transforms on $L^p(\mathbb{R})$, for $1 < p < \infty$.

Theorem 3.1. *Let $f \in L^p(\mathbb{R})$, for $1 < p < \infty$. Then, for any given $y > 0$,*

(i) $\int\limits_{-\infty}^{\infty} |U(x,y)|^p \, dx \leq \int\limits_{-\infty}^{\infty} |f(x)|^p \, dx$

(ii) $\int\limits_{-\infty}^{\infty} |V(x,y)|^p \, dx \leq A_p \int\limits_{-\infty}^{\infty} |f(x)|^p \, dx$

(iii) $\int\limits_{-\infty}^{\infty} |F(x+iy)|^p \, dx \leq A_p \int\limits_{-\infty}^{\infty} |f(x)|^p \, dx$

where A_p is a positive constant dependent only on p.

Proof. Since $P_y \in L^1(\mathbb{R})$ with $\|P_y(x)\|_1 = 1$ and $f \in L^p(\mathbb{R})$, we use Young's theorem (Theorem 6.3) and have $\|U(x,y)\|_p = \|(f*P_y)(x)\|_p \leq \|f\|_p\|P_y\|_1 = \|f\|_p$.

The proof of $\int\limits_{-\infty}^{\infty} |V(x,y)|^p \, dx \leq A \int\limits_{-\infty}^{\infty} |f(x)|^p \, dx$, for some $A > 0$, consists of proving the result first for $1 < p \leq 2$, and then applying a duality argument to prove that the result holds for $2 < p < \infty$ as well. We suggest the reader to refer to [5] for details.

Using the above two facts and using Minkowski's inequality, we have $\|F(z)\|_p = \|U(x,y) + iV(x,y)\|_p \leq \|U(x,y)\|_p + \|V(x,y)\|_p \leq B\|f\|_p$, for some $B > 0$.

Choice of A_p can be made from A and B appropriately. $\qquad \square$

Theorem 3.2. *Let $f \in L^p(\mathbb{R})$, for $1 < p < \infty$. Then,*

(i) *for each $\varepsilon > 0$, $\|\tilde{f}_\varepsilon\|_p \leq A_p\|f\|_p$*

(ii) *there exists $\tilde{f} \in L^p(\mathbb{R})$ such that $\|\tilde{f}_\varepsilon - \tilde{f}\|_p \to 0$ as $\varepsilon \to 0$*

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(iii) $\|\tilde{f}\|_p \leq A_p \|f\|_p$

where A_p is a positive constant dependent only on p.

Proof. We define $K(t) = \frac{t}{1+t^2} - \frac{1}{t}\mathcal{X}_{|t|>1}$ so that we have $K_\varepsilon(t) = \frac{1}{\varepsilon}K(\frac{t}{\varepsilon}) = \frac{t}{\varepsilon^2+t^2} - \frac{1}{t}\mathcal{X}_{|t|>\varepsilon}$. We observe that $K_\varepsilon(t)$ is an integrable kernel (even though it is difference of two non-integrable kernels). Moreover, $\int\limits_{-\infty}^{\infty} K(t)\,dt = 0$, since $K(t)$ is odd.

Then, we recall our definition $V(x,y) = (f * \tilde{P}_y)(x)$.

Now,

$$V(x,\varepsilon) - \tilde{f}_\varepsilon(x) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(x-t)\frac{t}{\varepsilon^2+t^2}\,dt - \frac{1}{\pi}\int\limits_{|t|>\varepsilon} f(x-t)\frac{1}{t}\,dt$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{\infty} f(x-t)\left[\frac{t}{\varepsilon^2+t^2} - \frac{\mathcal{X}_{|t|>\varepsilon}}{t}\right]\,dt$$

$$= \frac{1}{\pi}(f * K_\varepsilon)(x)$$

From Young's theorem (Theorem 6.3), we have $\|V(x,\varepsilon) - \tilde{f}_\varepsilon(x)\|_p \leq \frac{\|K\|_1}{\pi}\|f\|_p$.

On the other hand, (Theorem 3.1) showed that $\|V(x,\varepsilon)\|_p \leq A_p\|f\|_p$.

Then, using Minkowski's inequality, we have
$\|\tilde{f}_\varepsilon\|_p \leq \|\tilde{f}_\varepsilon - V(x,\varepsilon)\|_p + \|V(x,\varepsilon)\|_p \leq (A_p + \frac{\|K\|_1}{\pi})\|f\|_p$.

To prove the existence of $\tilde{f} \in L^p(\mathbb{R})$ such that $\|\tilde{f} - \tilde{f}_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$, we have to prove that $\{\tilde{f}_\varepsilon\}$ is a Cauchy sequence. The proof consists of proving first that $\{\tilde{g}_\varepsilon\}$ is a Cauchy sequence for each $g \in \mathcal{C}_0^1(\mathbb{R})$. Here, $\mathcal{C}_0^1(\mathbb{R})$ is the space of continuously differentiable functions with compact support. Since $\mathcal{C}_0^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, it follows that, in general for $f \in L^p(\mathbb{R})$, $1 < p < \infty$, $\{\tilde{f}_\varepsilon\}$ forms a Cauchy sequence. We suggest the reader to refer to [5] for details.

Finally, the fact that $\|\tilde{f}\|_p \leq A_p \|f\|_p$ follows from the first two facts of the theorem and Minkowski's inequality. $\qquad \square$

From the above theorem, it follows that Hilbert transform is a continuous linear functional on $L^p(\mathbb{R})$.

Theorem 3.3. *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then, $\tilde{f}_\varepsilon(x) \to \tilde{f}(x)$ pointwise a.e.*

Proof. By (Theorem 3.2), $\tilde{f} \in L^p(\mathbb{R})$. Recalling that $P_\varepsilon(t) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + t^2}$ and $\|P_\varepsilon\|_1 = 1$, by Young's theorem (Theorem 6.3) , we have that $(\tilde{f} * P_\varepsilon)(x)$ exists a.e. and $\|\tilde{f} * P_\varepsilon\|_p \leq \|\tilde{f}\|_p \|P_\varepsilon\|_1 = \|\tilde{f}\|_p$. Thus, $\tilde{f} * P_\varepsilon \in L^p(\mathbb{R})$.

By (Theorem 3.1), $V(x, \varepsilon) = (f * \tilde{P}_\varepsilon)(x) \in L^p(\mathbb{R})$. We claim that:

$$f * \tilde{P}_\varepsilon = \tilde{f} * P_\varepsilon \tag{11}$$

From (Corollary 2.3), we know that that (11) is true for all $f \in L^2(\mathbb{R})$. So, in particular, (11) is true for $f \in \mathcal{C}_0^1(\mathbb{R})$. Since $\mathcal{C}_0^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, it follows that (11) is true for all $f \in L^p(\mathbb{R})$, $1 < p < \infty$.

Then, from (Theorem 6.4), we have $(\tilde{f} * P_\varepsilon)(x) \to \tilde{f}(x)$ pointwise a.e. as $\varepsilon \to 0$. Thus, we have $(f * \tilde{P}_\varepsilon)(x) \to \tilde{f}(x)$ pointwise a.e. as $\varepsilon \to 0$.

Further, using kernel K_ε as defined in the beginning of proof of (Theorem 3.2) and arguing similarly, we have: $(f * \tilde{P}_\varepsilon)(x) - \tilde{f}_\varepsilon(x) = (f * K_\varepsilon)(x) \to f(x) \int\limits_{-\infty}^{\infty} K(t)\, dt = 0$ pointwise a.e. as $\varepsilon \to 0$.

The preceding two paragraphs imply $\tilde{f}_\varepsilon(x) \to \tilde{f}(x)$ pointwise a.e. as $\varepsilon \to 0$. $\qquad \square$

Corollary 3.4. *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then $\tilde{\tilde{f}} = -f$.*

Proof. We saw in (Theorem 2.1) that $\tilde{\tilde{f}} = -f$ is true for $f \in L^2(\mathbb{R})$. In particular, it is true for $f \in \mathcal{C}_0^1(\mathbb{R})$. The result for $f \in L^p(\mathbb{R})$, $1 < p < \infty$ follows since $C_0^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. $\qquad \square$

Thus, for $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have seen that $\tilde{f}_\varepsilon(x) = \frac{1}{\pi} \int\limits_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt \to$ $\tilde{f}(x)$ pointwise a.e. and in L^p as $\varepsilon \to 0$. And, the continuous linear functional $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is invertible with $H^{-1} = -H$.

4 Hilbert transforms on $L^1(\mathbb{R})$

Hilbert transform of an $L^1(\mathbb{R})$ function is not necessarily an $L^1(\mathbb{R})$ function. For example, while Poisson kernel $P_\varepsilon \in L^1(\mathbb{R})$, its Hilbert transform, the conjugate Poisson kernel $\tilde{P}_\varepsilon \notin L^1(\mathbb{R})$.

In this section, we give some results on Hilbert transforms of functions in $L^1(\mathbb{R})$.

Theorem 4.1. *Let $f \in L^1(\mathbb{R})$. Then, $\tilde{f}(x)$ exists a.e. Moreover, if $E(y) = |\{x : |\tilde{f}(x)| > y > 0\}|$ is the distribution function of $\tilde{f}$, then $E(y) \leq A\frac{\|f\|_1}{y}$, where $A > 0$ is a constant independent of y and f.*

The proof of the above theorem follows from a result in real analysis that involves derivates of functions of bounded variation. [5] gives the details.

Theorem 4.2. *Let $f_n \in L^1(\mathbb{R})$ and $f_n \to f$ in L^1. Then, $\tilde{f}_n \to \tilde{f}$ in measure. And, for some subsequence $\{f_{n_k}\}$, $\tilde{f}_{n_k} \to \tilde{f}$ pointwise a.e.*

Proof. Applying (Theorem 4.1) to $f_n - f$, we have that for every $\varepsilon > 0$, $|\{x : |\tilde{f}_n(x) - \tilde{f}(x)| > \varepsilon\}| \leq \frac{A}{\varepsilon}\|f_n - f\|_1 \to 0$ as $n \to \infty$.
Thus, $\tilde{f}_n \to \tilde{f}$ in measure.

From real analysis, we know that any sequence of functions that converges in measure has a subsequence that converges pointwise a.e. $\qquad\square$

5 Conclusion

In this paper, we studied the existence and some properties of Hilbert transforms of functions in $L^p(\mathbb{R}), 1 \leq p < \infty$. We first studied Hilbert transforms on $L^2(\mathbb{R})$ space using results such as Plancheral-Parseval theorem from Fourier analysis of $L^2(\mathbb{R})$ space. We then studied Hilbert transforms on $L^p(\mathbb{R})$ spaces for $1 < p < \infty$, where the results for $L^2(\mathbb{R})$ space Hilbert transforms proved useful. Finally, we gave some results for Hilbert transforms on $L^1(\mathbb{R})$ space.

Follow up work could be to study singular integrals which are a generalization of Hilbert transforms for functions on $\mathbb{R}^n$. Secondly, as an application of Hilbert transforms, the complex wavelet transforms could be studied in depth.

6 Appendix: Fourier Analysis

This appendix comprises of definitions and results in Fourier Analysis that were used in the prior sections.

Definition 6.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. Their convolution $f * g$ is defined as:*

$$(f * g)(x) = \int_{\mathbb{R}} f(t)g(x - t)\, dt$$

Theorem 6.2. *Let $f, g \in L^1(\mathbb{R})$. Then, $f * g$ exists a.e. and belongs to $L^1(\mathbb{R})$. Also, $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.*

Theorem 6.3 (W.H.Young). *Let $1 \le p \le \infty$. Let $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then, $f * g$ exists a.e. and belongs to $L^p(\mathbb{R})$. Also, $\|f * g\|_p \le \|f\|_p \|g\|_1$.*

Theorem 6.4. *Let $f \in L^p(\mathbb{R}), 1 \le p < \infty$. Let $P_\varepsilon(x)$ denote the Poisson kernel $\frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + x^2}$. Then, $f * P_\varepsilon$ converges to f pointwise a.e. and in the L^p norm as $\varepsilon \to 0$.*

Definition 6.5. *Let $f \in L^1(\mathbb{R})$. The Fourier transform, denoted $\hat{f}$, of f is defined as:*

$$\hat{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(t)e^{-ixt}\, dt$$

It is clear that $\hat{f}$ exists for each $f \in L^1(\mathbb{R})$.

Theorem 6.6. *Let $f, g \in L^1(\mathbb{R})$ and $h = f * g$. Then, $\hat{h} = \hat{f}\hat{g}$.*

Definition 6.7. *Let $f \in L^2(\mathbb{R})$. The Fourier transform $\hat{f}(x)$ is defined as the limit of truncated integrals $\frac{1}{2\pi} \int_{|t|<M} f(t)e^{-ixt}\, dt$ in $L^2(\mathbb{R})$ as $M \to \infty$.*

Theorem 6.8 (Parseval-Plancheral). *Let $f \in L^2(\mathbb{R})$. Then,*

(i) $\hat{f}$ exists.

(ii) $\|\hat{f}\|_2 = \frac{1}{2\pi} \|f\|_2$ (Parseval formula)

(iii) $\int\limits_{|t| \leq M} \hat{f}(t)e^{ixt}\, dt \to f(x)$ *in* $L^2(\mathbb{R})$ *as* $M \to \infty$.

(iv) $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *such that* $F(f) = \hat{f}$ *is a linear bijection.*

Theorem 6.9. *Let* $f \in L^2(\mathbb{R})$, $g \in L^1(\mathbb{R})$ *and* $h = f * g$. *Then,* $\hat{h} = \hat{f}\hat{g}$ *a.e.*

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